

# Uniqueness for a hyperbolic inverse problem with angular control on the coefficients

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## Abstract

Suppose  $q_i(x)$ ,  $i = 1, 2$  are smooth functions on  $\mathbb{R}^3$  and  $U_i(x, t)$  the solutions of the initial value problem

$$\begin{aligned}\partial_t^2 U_i - \Delta U_i - q_i(x)U_i &= \delta(x, t), & (x, t) \in \mathbb{R}^3 \times \mathbb{R} \\ U_i(x, t) &= 0, & \text{for } t < 0.\end{aligned}$$

Pick  $R, T$  so that  $0 < R < T$  and let  $C$  be the vertical cylinder  $\{(x, t) : |x| = R, R \leq t \leq T\}$ . We show that if  $(U_1, U_{1r}) = (U_2, U_{2r})$  on  $C$  then  $q_1 = q_2$  on the annular region  $R \leq |x| \leq (R+T)/2$  provided there is a  $\gamma > 0$ , independent of  $r$ , so that

$$\int_{|x|=r} |\Delta_S(q_1 - q_2)|^2 dS_x \leq \gamma \int_{|x|=r} |q_1 - q_2|^2 dS_x, \quad \forall r \in [R, (R+T)/2].$$

Here  $\Delta_S$  is the spherical Laplacian on  $|x| = r$ .

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# 1 Introduction

Our goal is the study of a formally determined inverse problem for a hyperbolic PDE. Consider an acoustic medium, occupying the region  $\mathbb{R}^3$ , excited by an impulsive point source and the response of the medium is measured for a certain time period at receivers placed on a sphere surrounding the source. We study the question of recovering the acoustic property of the medium from this measurement.

Let  $q(x)$  be a smooth function on  $\mathbb{R}^3$  and  $U(x, t)$  the solution of the initial value problem

$$U_{tt} - \Delta U - q(x)U = 8\pi\delta(x, t), \quad (x, t) \in \mathbb{R}^3 \times \mathbb{R}, \quad (1.1)$$

$$U = 0, \quad t < 0. \quad (1.2)$$

Using the progressing wave expansion one may show that

$$U(x, t) = 2 \frac{\delta(t - |x|)}{|x|} + u(x, t)H(t - |x|), \quad (1.3)$$

where  $u(x, t)$  is the solution of the Goursat problem

$$u_{tt} - \Delta u - q(x)u = 0, \quad (x, t) \in \mathbb{R}^3, \quad t \geq |x|, \quad (1.4)$$

$$u(x, |x|) = \int_0^1 q(\sigma x) d\sigma. \quad (1.5)$$

The well posedness of the above Goursat problem is proved in [9] and improved in [11], though the result is not optimal; [9] has suggestions for obtaining better results and we will address them elsewhere. For completeness we restate the well posedness result.

**Theorem 1.1** (See [9] and [11]). *Suppose  $\rho > 0$ , and  $q$  is a  $C^8$  function on the ball  $|x| \leq \rho$ ; then (1.4), (1.5) has a unique  $C^2$  solution on the double conical region  $\{(x, t) \in \mathbb{R}^3 \times \mathbb{R} : |x| \leq \rho, |x| \leq t \leq 2\rho - |x|\}$ . Further, the  $C^2$  norm of  $u$ , on this double conical region, approaches zero if the  $C^8$  norm of  $q$ , on  $|x| \leq \rho$ , approaches zero. Also, if  $q$  is smooth then so is  $u$ .*

Below  $P \preccurlyeq Q$  will mean that  $P \leq CQ$  for some constant  $C$ . Let  $S$  denote the unit sphere centered at the origin. For any  $0 < R < T$ , we define (see Figure 1) the annular region

$$A := \{x \in \mathbb{R}^3 : R \leq |x| \leq (R + T)/2\},$$

the space-time cylinder

$$C = \{(x, t) \in \mathbb{R}^3 \times \mathbb{R} : |x| = R, R \leq t \leq T\},$$

and

$$K := \{(x, t) \in \mathbb{R}^3 \times \mathbb{R} : R \leq |x| \leq (R + T)/2, |x| \leq t \leq R + T - |x|\},$$

a region bounded by  $C$  and two light cones.

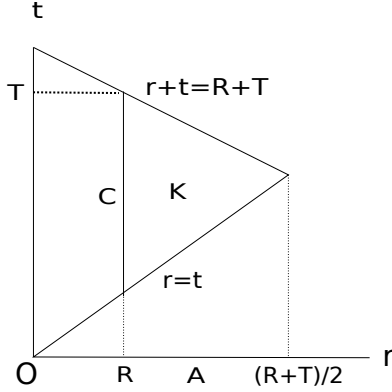


Figure 1: Geometry of the problem

In our model the source is at the origin, the receivers are on the sphere  $|x| = R$  and the signals are measured up to time  $T$ . Hence we define the forward map

$$F : q \mapsto (u|_C, u_r|_C)$$

and our goal is to study the injectivity and the inversion of  $F$ . From the domain of dependence property of solutions of hyperbolic PDEs, it is clear that  $F(q)$  is unaffected by changes in  $q$  in the region  $|x| \geq (R+T)/2$ . Hence the best we can hope to do is recover  $q$  on the ball  $|x| \leq (R+T)/2$ .

If  $q$  is spherically symmetric then the problem reduces to an inverse problem for the one dimensional wave equation. In this case, recovering  $q$  on the region  $R \leq |x| \leq (R+T)/2$ , from  $F(q)$ , is done by the downward continuation method or the layer stripping method - see [16] and other references there. However, even in the spherically symmetric case (i.e. the one dimensional case), recovering  $q$  on  $|x| \leq R$ , from  $F(q)$  is more difficult since the downward continuation scheme is not directly applicable. It is believed that uniqueness does not hold for this inverse problem if  $T < 3R$  though explicit examples have not been constructed. If  $T \geq 3R$ , the question of recovering  $q$  on  $|x| \leq R$  from  $(u, u_r)|_C$  was resolved by connecting this problem to one where the downward continuation method is applicable - see [8] and the references there. So it seems that in the general  $q$  case, recovering  $q$  over the region  $|x| \leq R$  will be harder than recovering  $q$  over the region  $R \leq |x| \leq (R+T)/2$ .

Our main result concerns the problem of recovering  $q$  on  $R \leq |x| \leq (R+T)/2$  from  $(u, u_r)|_C$ . The downward continuation method does not apply directly in higher space dimensions since the time-like Cauchy problem for hyperbolic PDEs is ill-posed in higher space dimensions. Further, an analysis of the linearized problem shows that there could be singularities in  $q$  in certain directions, that is points in the wave front set of  $q$ , so that a signal emanating from the origin is reflected by this singularity in  $q$ , and the reflected signal never reaches the sphere  $|x| = R$  where the receivers are located - see Figure 2. Hence there should not be any stability for this inverse problem, unless we restrict  $q$  to a class of functions where singularities in  $q$  of the above type are controlled. In [13], Sacks and Symes adapted the downward continuation method to apply to a slightly different inverse

problem, with an impulsive planar source  $\delta(z - t)$ , with data measured on the hypersurface  $z = 0$ , where  $x = (y, z)$  with  $y \in \mathbb{R}^2$  and  $z \in \mathbb{R}$ . They proved uniqueness for the linearized inverse problem when the unknown coefficient was restricted to the class of functions whose derivatives in the  $y$  direction were controlled by derivatives in the  $z$  direction. Later Romanov showed the inversion methods for one dimensional problems could be used for the existence and reconstruction for the nonlinear version of the Sacks and Symes inverse problem provided  $q(y, z)$  lies in the class of functions which are analytic in  $y$  in a certain sense, that is strong restrictions are placed on the changes in  $q$  in the  $y$  direction - see [10] for details. We apply the technique in [13] to the uniqueness question for the problem of recovering  $q$  on  $R \leq |x| \leq (R + T)/2$  from  $(u, u_r)|_C$ ; we will have to impose restrictions on the angular derivatives of  $q$ .

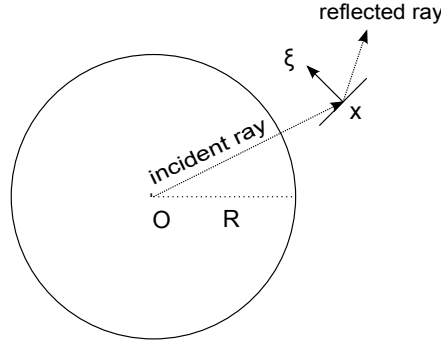


Figure 2: Reflection by a singularity in  $q$

For any  $x \in \mathbb{R}^3$  we define  $r = |x|$  and for  $x \neq 0$  we define  $\theta = x/r \in S$ ; hence  $x = r\theta$ . Define the radial vector field  $\partial_r = r^{-1}x \cdot \nabla$  and, for  $1 \leq i < j \leq 3$ , the angular vector fields  $\Omega_{ij} = x_i\partial_j - x_j\partial_i$ .

**Definition 1.2.** Given  $\gamma > 0$ , we define  $Q_\gamma(R, T)$  to be the set of all  $C^2$  functions  $q(x)$  on the ball  $|x| \leq (R + T)/2$  with

$$\|p\|_{H^2(S_r)} + \|\partial_r p\|_{H^1(S_r)} \leq \gamma (\|p\|_{H^1(S_r)} + \|\partial_r p\|_{L^2(S_r)}) \quad \forall r \in [R, (R + T)/2]$$

where  $p(x) = \int_0^{|x|} q(\sigma x/|\sigma|) d\sigma$  and  $S_r$  is the sphere  $|x| = r$ .

So if  $q$  is a smooth function on  $|x| \leq (R + T)/2$  with  $\|p\|_{H^1(S_r)} + \|\partial_r p\|_{L^2(S_r)}$  nonzero for every  $r \in [R, (R + T)/2]$  then  $q \in Q_\gamma$  where

$$\gamma = \max_{r \in [R, (R+T)/2]} \frac{\|p\|_{H^2(S_r)} + \|\partial_r p\|_{H^1(S_r)}}{\|p\|_{H^1(S_r)} + \|\partial_r p\|_{L^2(S_r)}}.$$

Noting that  $\partial_r p = q$ , using Garding's inequality on a sphere<sup>1</sup>, one may show that  $q \in Q_{\gamma^*}$  for some  $\gamma^* > 0$  if there is a  $\gamma > 0$  so that

$$\|\Delta_S q\|_{L^2(S_r)} \leq \gamma \|q\|_{L^2(S_r)}, \quad \forall r \in [R, (R + T)/2]$$

<sup>1</sup>The Euclidean version is (6.8) on page 66 of [4]. Using a partition of unity argument and the Euclidean version, one may show that  $\|q\|_{H^2(S_r)} \leq C_r \|\Delta_S q\|_{L^2(S_r)}$  with  $C_r$  bounded if  $r$  is in a closed interval not containing 0.

where  $\Delta_S$  is the Laplacian on  $S_r$ . In particular, if  $q$  is a finite linear combination of the spherical harmonics with coefficients dependent on  $r$  then  $q \in Q_\gamma(R, T)$  for some  $\gamma > 0$ .

In section 2 we prove the following injectivity result using the ideas in [13].

**Theorem 1.3.** *Suppose  $0 < R < T$  and  $q_1, q_2$  are  $C^8$  functions on  $\mathbb{R}^3$ . If  $F(q_1) = F(q_2)$  and  $q_1 - q_2 \in Q_\gamma(R, T)$  for some  $\gamma > 0$  then  $q_1 = q_2$  on  $R \leq |x| \leq (R + T)/2$ .*

One may tackle the problem dealt with in Theorem 1.3 using Carleman estimates also and one obtains a result which is stronger in some aspects and weaker in others. Using Carleman estimates one can prove uniqueness under slightly less stringent conditions on  $q$  - one needs controls on the  $L^2$  norms of only the first order angular derivatives of  $p$  in terms of the  $L^2$  norm of  $p$ , instead of on the second order angular derivatives required in Theorem 1.3. However, the price one pays is that the  $\gamma$  cannot be arbitrary but is determined by  $R, T$ ; further  $R$  cannot be arbitrary, but must satisfy  $R > T/2$  and uniqueness is proved only for the values of  $q$  in an annular region  $R \leq |x| \leq R^*$  for some  $R^* < (R + T)/2$ . This work will appear elsewhere.

From Theorem 1.3 we can easily derive the following interesting corollary.

**Corollary 1.4.** *Suppose  $0 < T$  and  $q_1, q_2$  are smooth functions on  $\mathbb{R}^3$  which vanish in a neighborhood of the origin. If  $u_1$  and  $u_2$  agree to infinite order on the line  $\{(x = 0, t) : 0 \leq t \leq T\}$  and  $q_1 - q_2 \in Q_\gamma(0, T)$  for some  $\gamma > 0$ , then  $q_1 = q_2$  on  $|x| \leq T$ .*

We give a short proof of the corollary. If  $q_1 = q_2 = 0$  in some small neighborhood of the origin then the difference  $u = u_1 - u_2$  satisfies the standard homogeneous wave equation in a semi-cylindrical region

$$\{(x, t) \in \mathbb{R}^3 \times \mathbb{R} : |x| \leq \delta, |x| \leq t \leq T - |x|\}, \quad (1.6)$$

for some  $\delta > 0$ . Now, from the hypothesis, we have  $u$  is zero to infinite order on the segment of the  $t$  axis consisting of  $0 \leq t \leq T$ . Then by Lebeau's unique continuation result in [5] we have  $u = 0$  in the semi-cylindrical region given in (1.6). Hence  $u$  and  $u_r$  are zero on the cylinder

$$\{(x, t) \in \mathbb{R}^3 \times \mathbb{R} : |x| = \delta, \delta \leq t \leq T - \delta\}.$$

The corollary follows from Theorem (1.3) if the  $R$  and  $T$  in Theorem 1.3 are taken to be  $\delta$  and  $T - \delta$  respectively.

We also have a uniqueness result for the linearized version of the inverse problem considered in Theorem 1.3; the result is for a linearization about a radial background.

**Theorem 1.5.** *Suppose  $q_b(r)$  is a function on  $[0, \infty)$  so that  $q_b(|x|)$  is a smooth function on  $\mathbb{R}^3$ ; further suppose  $u_b(r, t)$  is the solution of (1.4), (1.5) when  $q(x)$  is replaced by  $q_b(|x|)$ . Let  $q(x)$  be*

a smooth function on  $\mathbb{R}^3$  and  $u(x, t)$  the solution of the Goursat problem

$$u_{tt} - \Delta u - q_b u = q u_b, \quad t \geq |x|, \quad (1.7)$$

$$u(x, |x|) = \int_0^1 q(\sigma x) d\sigma. \quad (1.8)$$

If  $(u, u_r)|_C = 0$  then  $q = 0$  on the region  $R \leq |x| \leq (R + T)/2$ .

This theorem holds with less regular  $q_b$  and  $q$ ; what is needed is enough regularity so that the spherical harmonic expansions of  $q$ ,  $q_b$  and  $u_b$  converge in the  $C^2$  norm.

We next focus on the problem of recovering  $q$  on the region  $|x| \leq R$  from  $(u, u_r)|_C$  when  $T \geq 3R$ . The linearized problem about the  $q = 0$  background, consisting of recovering  $q$  from  $(u, u_r)|_C$ , where  $u(x, t)$  is the solution of the Goursat problem

$$u_{tt} - \square u = 0, \quad t \geq |x|, \\ u(x, |x|) = \int_0^1 q(\sigma x) d\sigma.$$

As observed by Romanov, since  $T \geq 3R$ , we may recover  $q$  from  $(u, u_r)|_C$  fairly quickly. In fact, from Kirchhoff's formula (see [2]) expressing the solution of the wave equation in terms of the Cauchy data on  $C$ , we have

$$u(x, t) = \int_{|y-x|=R} \frac{u_r(y, t + |x - y|)}{|x - y|} + \left( \frac{u(y, t + |x - y|)}{|x - y|^2} + \frac{u_t(y, t + |x - y|)}{|x - y|} \right) \frac{(y - x) \cdot y}{|x - y|} dS_y.$$

for all  $(x, t)$  with  $|x| \leq t \leq R$  - see Figure 3. In particular we can express  $u(x, |x|)$  in terms of  $(u, u_t, u_r)|_C$  and hence we can recover  $q$ .

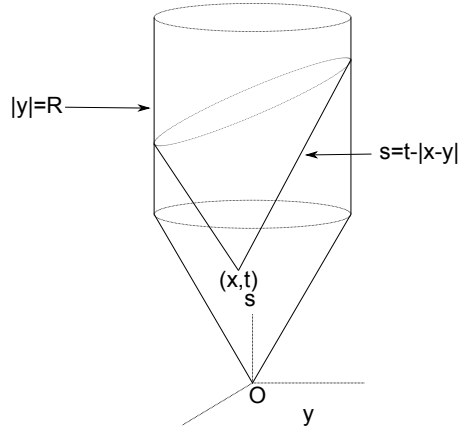


Figure 3: Kirchhoff's Formula

For the *original nonlinear inverse problem* we show a partial uniqueness and stability result when one of the  $q$  is small.

**Theorem 1.6.** Suppose  $0 < 3R < T$ ,  $M > 0$  and  $q_i$ ,  $i = 1, 2$  are  $C^8$  functions on  $|x| \leq (R + T)/2$  with  $\|q_i\|_\infty \leq M$ . Let  $u_i$  be the unique solution of (1.4), (1.5) with  $q$  replaced by  $q_i$ ; then there is a constant  $\delta > 0$  depending only on  $R, T$  and  $M$  so that if  $\|q_2\|_\infty \leq \delta$  then

$$\int_{|x| \leq R} |q_1 - q_2|^2 dx \preceq \int_C |u_1 - u_2|^2 + |\nabla(u_1 - u_2)|^2 + |(u_1 - u_2)_t|^2 dS_{x,t}; \quad (1.9)$$

the constant in (1.9) depending only on  $R, T, M$ .

A weaker form of this result, requiring that  $\|q_1\| \leq \delta$  also, was given in [7]; a result similar to this weaker version was also derived in [12]. Later it was observed in [6], for a similar type of problem, that the above proofs go through without the extra assumption that  $\|q_1\| \leq \delta$ . We give this short proof of Theorem 1.6, in section 4. However, the original nonlinear inverse problem remains unsolved.

## 2 Proof of Theorem 1.3

### 2.1 Preliminary observations

We need the following observations in the proof. For the angular vector fields we have  $[\Omega_{ij}, \partial_r] = 0$ , and  $[\Omega_{ij}, \Omega_{kl}] = 0$  if  $\{i, j\} = \{k, l\}$  but  $[\Omega_{ij}, \Omega_{ik}] = \Omega_{kj}$ . Also  $|\nabla f|^2 = f_r^2 + r^{-2} \sum_{i < j} (\Omega_{ij} f)^2$  and if we define  $\Omega = \sum_{i < j} \Omega_{ij}^2$  then  $\Delta = \partial_r^2 + 2r^{-1} \partial_r + r^{-2} \Omega$  and  $[\Omega_{ij}, \Delta] = 0$ . Also, for any  $i \neq j$ , since  $\Omega_{ij} f = x_i \partial_j f - x_j \partial_i f = \partial_j(x_i f) - \partial_i(x_j f)$  and  $x_j x_i - x_i x_j = 0$ , by the divergence theorem, for any  $0 < R_1 < R_2$  we have

$$\int_{R_1 \leq |x| \leq R_2} \Omega_{ij} f dx = 0. \quad (2.1)$$

Applying (2.1) to the zeroth order homogeneous extension of a function  $f$  on  $S$ , we conclude that for  $C^1$  functions  $f, g$  on  $S$

$$\int_S \Omega_{ij} f dS = 0, \quad \int_S f \Omega_{ij} g dS = - \int_S g \Omega_{ij} f dS. \quad (2.2)$$

For  $i = 1, 2$  let  $u_i$  be the solution of (1.4), (1.5) when  $q = q_i$ . Define  $v_i(x, t) = r u_i(x, t)$ ,  $p_i(x) = r \int_0^1 q_i(\sigma x) d\sigma = \int_0^r q_i(\sigma \theta) d\sigma$ . Define  $v = v_1 - v_2$ ,  $q = q_1 - q_2$  and  $p = p_1 - p_2$ . Then we have

$$v_{tt} - v_{rr} - \frac{1}{r^2} \Omega v - q_1 v = q v_2, \quad t \geq |x| \quad (2.3)$$

$$v(x, |x|) = p(x). \quad (2.4)$$

We are given that  $(v, v_r)$  are zero on  $C$  and we have to show that  $q = 0$  on  $R \leq |x| \leq (R + T)/2$ . Note that since  $v = 0$  on  $C$ , we have  $p(x) = v(x, |x|) = 0$  on  $|x| = R$  and hence for  $|x| \geq R$  we have  $p(x) = \int_R^r q(\sigma\theta) d\sigma$  and hence to prove the theorem it will be enough to show that  $p(x) = 0$  on  $R \leq |x| \leq (R + T)/2$ .

We will attempt to carry out the proof which works in the one dimensional case. The limitations of this method when applied to the three dimensional case force the restrictions on  $q$  in the statement of Theorem 1.3. In the one dimensional case the angular terms are missing from (2.3) so the roles of  $r, t$  are reversible and one has sideways energy estimates which allow us to estimate the  $H^1$  norm of  $v$  on  $t = |x|$  in terms of the norm of  $v, v_r$  on  $r = R$  and the  $L^2$  norm of the RHS of (2.3). The  $H^1$  norm of  $v$  on  $t = |x|$  dominates the  $L^2$  norm of  $q$  on  $A$  and the  $L^2$  norm of the RHS of the (2.3) is dominated by  $T - R$  times the  $L^2$  norm of  $q$  on  $A$ . So if  $T - R$  is small enough we obtain  $q = 0$  on  $A$ ; then one combines a unique continuation argument with a repeated application of the above to prove that  $q = 0$  on  $A$  no matter what the  $T$ .

In the multidimensional case the above argument breaks down because of the angular Laplacian in (2.3); all other parts of the argument work as in the one dimensional case. To carry out the above procedure we will need two estimates. The first is a standard energy estimate for the wave equation and the second is an imitation of a sideways energy estimate for a one dimensional wave equation in  $r, t$  where the roles of  $r$  and  $t$  are reversed.

## 2.2 Energy identities

For each  $\rho \in [R, (R + T)/2]$ , define (see Figure 4) the sub-region

$$K_\rho := \{(x, t) \in \mathbb{R}^3 \times \mathbb{R} : R \leq |x| \leq \rho, |x| \leq t \leq R + T - |x|\},$$

the annular region

$$A_\rho := \{x \in \mathbb{R}^3 : R \leq |x| \leq \rho\},$$

the vertical cylinder

$$C_\rho := \{(x, t) \in \mathbb{R}^3 \times \mathbb{R} : |x| = \rho, \rho \leq t \leq R + T - \rho\},$$

and for any function  $w(x, t)$  let  $\bar{w}$  and  $\bar{\bar{w}}$  be the the restrictions of  $w$  to the lower and upper characteristic cones, that is

$$\bar{w}(x) = w(x, |x|), \quad \bar{\bar{w}}(x) = w(x, R + T - |x|).$$

We derive some relations which lead to the estimates we need. These relations are either the standard energy identity or a sideways version of it. Suppose  $w(x, t)$  satisfies

$$w_{tt} - w_{rr} - \frac{1}{r^2} \Omega w = F(x, t), \quad (x, t) \in K.$$



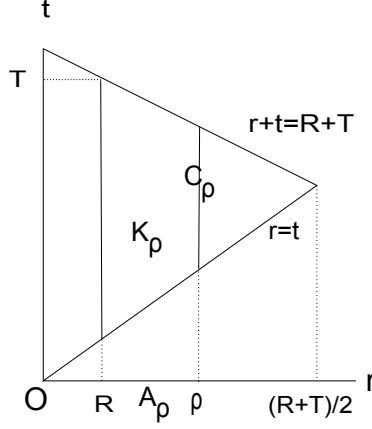


Figure 4: Sideways energy estimates

Define the “sideways” energy (we will assume a sum over  $1 \leq i < j \leq 3$ )

$$\begin{aligned}
 J(\rho) &:= \int_{C_\rho} r^{-2}(w^2 + w_t^2 + |\nabla w|^2) dS_{x,t} = \int_{C_\rho} r^{-2}(w_t^2 + w_r^2 + w^2 + r^{-2}(\Omega_{ij}w)^2) dS_{x,t} \\
 &= \int_\rho^{R+T-\rho} \int_S (w_t^2 + w_r^2 + w^2 + r^{-2}(\Omega_{ij}w)^2)(\rho\theta, t) d\theta dt.
 \end{aligned}$$

Multiplying the identity

$$\begin{aligned}
 2w_r(w_{tt} - w_{rr} - r^{-2}\Omega w - w) - 4r^{-2}\Omega_{ij}w_r\Omega_{ij}w + 2r^{-3}(\Omega_{ij}w)^2 \\
 = -(w_t^2 + w_r^2 + r^{-2}(\Omega_{ij}w)^2 + w^2)_r + 2(w_rw_t)_t - 2\Omega_{ij}(r^{-2}w_r\Omega_{ij}w)
 \end{aligned} \tag{2.5}$$

by  $r^{-2}$ , integrating over the region  $K_\rho$ , using (2.2) and Stokes’s theorem on a region in the  $r, t$

plane, we obtain

$$\begin{aligned}
& \int_{K_\rho} r^{-2} (2w_r(F-w) - 4r^{-2}\Omega_{ij}w_r\Omega_{ij}w + 2r^{-3}(\Omega_{ij}w)^2) dx dt \\
&= \int_S \int_R^\rho \int_r^{R+T-r} -(w_t^2 + w_r^2 + r^{-2}(\Omega_{ij}w)^2 + w^2)_r + 2(w_rw_t)_t dt dr d\theta \\
&= \int_S \int_R^T (w_t^2 + w_r^2 + r^{-2}(\Omega_{ij}w)^2 + w^2)(R\theta, t) dt d\theta \\
&\quad - \int_S \int_\rho^{R+T-\rho} (w_t^2 + w_r^2 + r^{-2}(\Omega_{ij}w)^2 + w^2)(\rho\theta, t) dt d\theta \\
&\quad - \int_S \int_R^\rho (w_t^2 + w_r^2 + r^{-2}(\Omega_{ij}w)^2 + w^2 - 2w_rw_t)(r\theta, R+T-r) dr d\theta \\
&\quad - \int_S \int_R^\rho (w_t^2 + w_r^2 + r^{-2}(\Omega_{ij}w)^2 + w^2 + 2w_rw_t)(r\theta, r) dr d\theta \\
&= J(R) - J(\rho) - \int_{A_\rho} r^{-2}(\bar{w}_r^2 + r^{-2}(\Omega_{ij}\bar{w})^2 + \bar{w}^2)(x) dx \\
&\quad - \int_{A_\rho} r^{-2}(\bar{w}_r^2 + r^{-2}(\Omega_{ij}\bar{w})^2 + \bar{w}^2)(x) dx.
\end{aligned}$$

Hence

$$\begin{aligned}
& J(\rho) + \int_{A_\rho} r^{-2}(|\nabla \bar{w}|^2 + \bar{w}^2)(x) dx + \int_{A_\rho} r^{-2}(|\nabla \bar{w}|^2 + \bar{w}^2)(x) dx + \int_{K_\rho} 2r^{-5}(\Omega_{ij}w)^2 dx dt \\
&= J(R) + \int_{K_\rho} r^{-2} (2ww_r + 4r^{-2}\Omega_{ij}w_r\Omega_{ij}w - 2Fw_r) dx dt, \quad R \leq \rho \leq \frac{R+T}{2}. \quad (2.6)
\end{aligned}$$

This is the sideways energy identity we need.

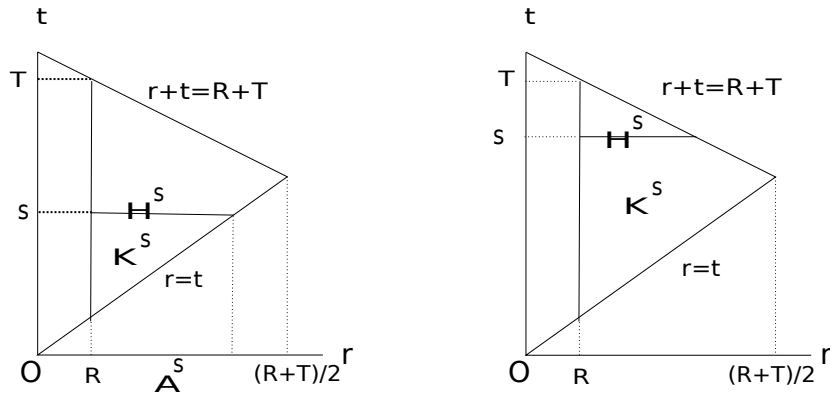


Figure 5: Standard energy estimate

Next we derive the standard energy identity for the wave equation. For any  $s \in [R, T]$ , define (see Figure 5) the domain

$$K^s = K \cap \{(x, t) \in \mathbb{R}^3 \times \mathbb{R} : R \leq t \leq s\},$$

$H^s$  the horizontal disk obtained by intersecting  $K$  with the plane  $t = s$ , that is

$$H^s = K \cap \{t = s\},$$

whose projection onto the plane  $t = 0$  is the annular region

$$A^s := \{x \in \mathbb{R}^3 : R \leq |x| \leq \min(s, R + T - s)\}.$$

Next, we define the “energy at time  $s$ ” for every  $s \in [R, T]$  - the definition depends on  $s \leq (R+T)/2$  or not because the geometry changes - see Figure 5. For  $s \in [R, (R+T)/2]$ , we define (summation over  $1 \leq k < l \leq 3$ )

$$E(s) := \int_{A^s} r^{-2}(w^2 + w_t^2 + w_r^2 + r^{-2}(\Omega_{kl}w)^2)(x, s) dx$$

and for  $s \in [(R+T)/2, T]$  we define

$$E(s) := \int_{A^s} r^{-2}(w^2 + w_t^2 + |\nabla w|^2)(x, s) dx + \int_{R+T-s \leq |x| \leq (R+T)/2} r^{-2}(\bar{w}(x)^2 + |(\nabla \bar{w})(x)|^2) dx.$$

First take  $s \leq (R+T)/2$ ; multiplying the identity

$$2w_t(w_{tt} - w_{rr} - r^{-2}\Omega w + w) = (w^2 + w_t^2 + w_r^2 + r^{-2}(\Omega_{kl}w)^2)_t - 2(w_t w_r)_r - 2r^{-2}\Omega_{kl}(w_t \Omega_{kl}w) \quad (2.7)$$

by  $r^{-2}$ , integrating over the region  $K^s$ , and using (2.1), we obtain

$$\begin{aligned} & \int_{K^s} 2r^{-2}w_t(F + w) dx dt \\ &= E(s) + 2 \int_R^s \int_{|x|=R} r^{-2}(w_t w_r)(x, s) dS_x dt \\ & \quad - \int_{A^s} r^{-2}(w^2 + w_t^2 + w_r^2 + 2w_t w_r + r^{-2}(\Omega_{kl}w)^2)(x, |x|) dx \\ &= E(s) + 2 \int_R^s \int_{|x|=R} r^{-2}(w_t w_r)(x, s) dS_x dt - \int_{A^s} r^{-2}(\bar{w}(x)^2 + |(\nabla \bar{w})(x)|^2) dx. \end{aligned}$$

Next take  $s \in [(R+T)/2, T]$ ; multiplying (2.7) by  $r^{-2}$ , integrating over the region  $K^s$ , using (2.1)

we obtain

$$\begin{aligned}
& \int_{K^s} 2r^{-2}w_t(F+w) dx dt \\
&= \int_{H^s} r^{-2}(w^2 + w_t^2 + |\nabla w|^2) dx \\
&\quad + \int_{R+T-s \leq |x| \leq (R+T)/2} r^{-2}(w^2 + w_t^2 + w_r^2 - 2w_t w_r + r^{-2}(\Omega_{kl}w)^2)(x, R+T-|x|) dx \\
&\quad - \int_A r^{-2}(w_t^2 + w_r^2 + 2w_t w_r + r^{-2}(\Omega_{kl}w)^2)(x, |x|) dx + 2 \int_R \int_{|x|=R}^s r^{-2}(w_t w_r)(x, s) dS_x dt \\
&= \int_{H^s} r^{-2}(w^2 + w_t^2 + |\nabla w|^2) dx + \int_{R+T-s \leq |x| \leq (R+T)/2} r^{-2}(\bar{w}(x)^2 + |(\nabla \bar{w})(x)|^2) dx \\
&\quad - \int_A r^{-2}(\bar{w}(x)^2 + |(\nabla \bar{w})(x)|^2) dx + 2 \int_R \int_{|x|=R}^s r^{-2}(w_t w_r)(x, t) dS_x dt \\
&= E(s) - \int_A r^{-2}(\bar{w}(x)^2 + |(\nabla \bar{w})(x)|^2) dx + 2 \int_R \int_{|x|=R}^s r^{-2}(w_t w_r)(x, t) dS_x dt.
\end{aligned}$$

Hence, in either case, that is for any  $s \in [R, T]$ , we have

$$E(s) \leq \int_A r^{-2}(\bar{w}(x)^2 + |(\nabla \bar{w})(x)|^2) dx + 2 \int_{K^s} r^{-2}w_t(F+w) dx dt + \int_C r^{-2}(w_t^2 + w_r^2) dS_x dt. \quad (2.8)$$

## 2.3 Uniqueness

We now show that if  $v$  and  $v_r$  are zero on  $C$  then  $q = 0$  on  $A$ . We apply (2.6) to  $v = v_1 - v_2$ ; note that  $F = v_{tt} - v_{rr} - r^{-2}\Omega v = q_1 v + q v_2$  and  $J(R) = 0$  because the Cauchy data of  $v$  is zero on  $C$ . Hence

$$\begin{aligned}
J(\rho) + \int_{A_\rho} r^{-2}(\bar{v}^2 + |\nabla \bar{v}|^2) &\leq \int_{K_\rho} r^{-2}(v v_r + 4r^{-2}\Omega_{ij}v_r \Omega_{ij}v - 2v_r(q_1 v + q v_2)) \\
&\preceq \int_{K_\rho} r^{-2}(v^2 + v_r^2 + r^{-2}(\Omega_{ij}v)^2 + q^2 + r^{-2}(\Omega_{ij}v_r)^2) \\
&= \int_R^\rho J(r) dr + \int_{A_\rho} r^{-2}q^2(x) \left( \int_r^{R+T-r} dt \right) dx + \int_{K_\rho} r^{-4}(\Omega_{ij}v_r)^2 \\
&\leq \int_R^\rho J(r) dr + (T-R) \int_A r^{-2}q^2(x) dx + \int_K r^{-4}(\Omega_{ij}v_r)^2
\end{aligned}$$

with the constant associated to  $\preceq$  being  $c_1 = 4 \max(1, \|q_1\|_{L^\infty(A)}, \|v_2\|_{L^\infty(K)})$ . Hence, by Gronwall's inequality

$$J(\rho) + \int_{A_\rho} r^{-2}(|\nabla p|^2 + p^2) \preceq (T-R) \int_A r^{-2}q^2(x) dx + \int_K r^{-4}(\Omega_{ij}v_r)^2, \quad R \leq \rho \leq \frac{R+T}{2}, \quad (2.9)$$

with the constant being  $c_2 = c_1 e^{c_1(T-R)}$ . In particular

$$J(\rho) \preccurlyeq (T-R) \int_A r^{-2} q^2(x) dx + \int_K r^{-4} (\Omega_{ij} v_r)^2, \quad R \leq \rho \leq \frac{R+T}{2}, \quad (2.10)$$

and taking  $\rho = (R+T)/2$  in (2.9) we have

$$\int_A r^{-2} (|\nabla p|^2 + p^2) \preccurlyeq (T-R) \int_A r^{-2} q^2(x) dx + \int_K r^{-4} (\Omega_{ij} v_r)^2 \quad (2.11)$$

with the constant  $c_2$ . Integrating (2.10) w.r.t  $\rho$  over  $[R, (R+T)/2]$  we obtain

$$\int_K r^{-2} (v^2 + v_t^2 + |\nabla v|^2) \preccurlyeq (T-R)^2 \int_A r^{-2} q^2(x) dx + (T-R) \int_K r^{-4} (\Omega_{ij} v_r)^2. \quad (2.12)$$

So we can combine (2.11), (2.12) into

$$\int_K r^{-2} (v^2 + v_t^2 + |\nabla v|^2) + \int_A r^{-2} (p^2 + |\nabla p|^2) \preccurlyeq (T-R) \int_A r^{-2} q^2 + \int_K r^{-4} (\Omega_{ij} v_r)^2 \quad (2.13)$$

with the constant being  $c_3 = (1+T-R)c_2$ .

The equation (2.13) would have been enough to prove Theorem 1 in the one dimensional case, because  $|\nabla p|^2 \geq p_r^2 = q^2$  and the last term in (2.13) would not be there. Then by taking  $T-R$  small enough we could have absorbed the second term on the RHS of (2.13) into the LHS and we would have proved the theorem for  $T$  close to  $R$ . Then a unique continuation argument would prove the theorem for all  $T > R$ . However, in the three dimensional case we do have the last term in (2.13) which cannot be absorbed in the LHS because it involves second order derivatives of  $v$  - we will estimate it in terms of  $p$  using the standard energy estimate for the wave operator.

Fix an  $i, j$  pair with  $i < j$ . We apply (2.8) to the function  $w = \Omega_{ij} v$ , noting that  $\Omega_{ij}$  commutes with  $\Omega$ . Note that from (2.3) and (2.4) we have

$$w_{tt} - w_{rr} - \frac{1}{r^2} \Omega w = F$$

with

$$F(x, t) := q_1 w + (\Omega_{ij} q_1) v + (\Omega_{ij} q) v_2 + q \Omega_{ij} v_2. \quad (2.14)$$

and

$$\bar{w}(x, |x|) = (\Omega_{ij} p)(x). \quad (2.15)$$

Further, since the Cauchy data of  $v$  is zero on  $C$ , so the Cauchy data of  $w$  is zero on  $C$ . Hence from (2.8) we have

$$\begin{aligned} E(s) &\leq \int_A r^{-2} ((\Omega_{ij} p)^2 + |\nabla \Omega_{ij} p|^2) + \int_{K^s} r^{-2} (w^2 + w_t^2 + F^2) \\ &\preccurlyeq \int_A r^{-2} ((\Omega_{ij} p)^2 + |\nabla \Omega_{ij} p|^2) + \int_{K^s} r^{-2} (w^2 + w_t^2 + v^2 + q^2 + (\Omega_{ij} q)^2) \\ &\preccurlyeq \int_R^s E(t) dt + \int_A r^{-2} (p^2 + |\nabla p|^2 + |\nabla \Omega_{ij} p|^2) + \int_K r^{-2} v^2 \end{aligned}$$

with the constant being  $c_4 = 2 \max(1, (R+T)^2, \|q_1\|_\infty, \|\Omega_{ij}q_1\|_\infty, \|v_2\|_\infty)$ . So from Gronwall's inequality we have

$$E(s) \preceq \int_A r^{-2}(p^2 + |\nabla p|^2 + |\nabla \Omega_{ij}p|^2) + \int_K r^{-2}v^2, \quad R \leq s \leq T$$

with the constant being  $c_5 = c_4 e^{c_4(T-R)}$ . Integrating this w.r.t  $s$  over the interval  $[R, T]$  we obtain

$$\int_K r^{-2}(w^2 + w_t^2 + |\nabla w|^2) \leq c_5(T-R) \left( \int_A r^{-2}(p^2 + |\nabla p|^2 + |\nabla \Omega_{ij}p|^2) + \int_K r^{-2}v^2 \right);$$

hence, since  $w = \Omega_{ij}v$ ,

$$\int_K r^{-4}(\Omega_{ij}v_r)^2 \leq c_5 R^{-2}(T-R) \left( \int_A r^{-2}(p^2 + |\nabla p|^2 + |\nabla \Omega_{ij}p|^2) + \int_K r^{-2}v^2 \right). \quad (2.16)$$

Using this in (2.13), we have

$$\begin{aligned} & \int_K r^{-2}v^2 + \int_A r^{-2}(p^2 + |\nabla p|^2) \\ & \preceq (T-R) \int_A r^{-2}(p^2 + |\nabla p|^2 + |\nabla \Omega_{ij}p|^2) + (T-R) \int_K r^{-2}v^2 \end{aligned} \quad (2.17)$$

with the constant  $c_6 = \max(c_3, c_3 c_5 R^{-2})$ . However,  $q$  is in  $Q_\gamma$  so

$$\begin{aligned} \int_A r^{-2} |\nabla(\Omega_{ij}p)(x)|^2 dx &= \int_R^{(R+T)/2} \int_{|\theta|=1} (\nabla \Omega_{ij}p)(r\theta)^2 d\theta dr \\ &\leq \gamma \int_R^{(R+T)/2} r^2 \int_{|\theta|=1} (p^2 + |\nabla p|^2)(r\theta) d\theta dr \\ &\leq \gamma(R+T)^2 \int_A r^{-2}(p^2 + |\nabla p|^2). \end{aligned}$$

Using this in (2.17), we see that  $p = 0$  on  $A$  if  $T-R$  is small enough - depending on  $\gamma, c_6$  and  $R+T$ . Now  $v(x, |x|) = p(x)$  and  $v = 0$  on  $|x| = R$  so  $p = 0$  on  $|x| = R$ , that is  $\int_0^R q(\sigma\theta) d\sigma = 0$  for all unit vectors  $\theta$ . Hence

$$\int_R^r q(\sigma\theta) d\sigma = 0, \quad R \leq r \leq T$$

which implies  $q(x) = 0$  when  $R \leq |x| \leq T$ , provided  $T-R$  is small enough.

Actually, adjusting the height of the downward pointing cone, what we have shown is the following: there is a  $\delta > 0$  dependent only on  $\gamma, R, T, \|q_1\|_{C^1(A)}, \|v_2\|_{C^1(K)}$ , so that if, for some  $R^* \in [R, (R+T)/2]$ ,  $v$  and  $v_r$  are zero on the cylinder

$$\{(x, t) : |x| = R^*, R^* \leq t \leq R^* + 2\delta\},$$

then  $q = 0$  on  $R^* \leq |x| \leq R^* + \delta$ , with the obvious modification in the assertion if  $R^* + \delta > (R+T)/2$ . We use this observation to prove that  $q = 0$  for any  $R, T$ .

Since  $v$  and  $v_r$  are zero on  $C$ , then from the above claim, we have  $q = 0$  on  $R \leq |x| \leq R + \delta$ . Let  $u = u_1 - u_2$  where  $u_1, u_2$  are solutions to (1.4), (1.5) for  $q = q_1, q_2$ . Then,  $u$  satisfies the homogeneous equation

$$u_{tt} - \Delta u - q_1 u = 0$$

over the region  $K_\rho$  where  $\rho = R + \delta$ . Now  $u$  and  $u_r$  are zero on  $C$ , and  $q_1$  is independent of  $t$ , so by the Robbiano-Tataru unique continuation theorem (see Theorem 3.16 in [KKL01]) we have  $u = 0$  in the region  $K_\rho$ ; in particular  $u$  and  $u_r$  are zero on  $C_\rho$  and hence  $v, v_r$  are zero on  $C_\rho$ . Now repeat the above argument, except  $R$  is replaced by  $R + \delta$ ; this argument repeated will complete the proof of Theorem 1.3.

### 3 Proof of Theorem 1.5

Let  $\{\phi_n(x)\}_{n=1}^\infty$  be a sequence of homogeneous harmonic polynomials on  $\mathbb{R}^3$  so that their restrictions to the unit sphere  $S$  form an orthonormal basis on  $L^2(S)$  - see Chapter 4 of [15]. Let  $k(n)$  be the degree of homogeneity of  $\phi_n$ . Then  $q(x)$  and  $u(x, t)$  have spherical harmonic decompositions in  $L^2(S)$  given by

$$q(r\theta) = \sum_{n=1}^{\infty} q_n(r) r^{k(n)} \phi_n(\theta), \quad u(r\theta, t) = \sum_{n=1}^{\infty} u_n(r, t) r^{k(n)} \phi_n(\theta)$$

where

$$r^{k(n)} q_n(r) = \int_{|\theta|=1} q(r\theta) \phi_n(\theta) d\theta, \quad r^{k(n)} u_n(r, t) = \int_{|\theta|=1} u(r\theta, t) \phi_n(\theta) d\theta.$$

Since  $u$  and  $q$  are smooth, we may show<sup>2</sup> that  $q_n(r)$  and  $u_n(r, t)$  decay as  $n^{-p}$  for large  $n$  for any positive integer  $p$ , uniformly in  $r, t$ . Hence the series also converge in the  $C^2$  norm.

To prove the theorem, it will be enough to prove that  $q_n(r) = 0$  on  $R \leq r \leq (R + T)/2$  for all  $n \geq 1$ . One may show that for sufficiently regular  $f$  (see page 1235 of [1])

$$\Delta \left( f(r, t) r^{k(n)} \phi_n(\theta) \right) = r^{k(n)} \phi_n(\theta) (f_{tt} - f_{rr} - \frac{2k(n) - 2}{r} f_r)$$

hence, using (1.7), (1.8), the  $u_n(r, t)$  are solutions of the one dimensional Goursat problems

$$\begin{aligned} \partial_t^2 u_n - \partial_r^2 u_n - \frac{2k(n) - 2}{r} \partial_r u_n - q_b u_n &= q_n u_b, \quad t \geq |r| \\ u_n(r, |r|) &= \int_0^1 \sigma^{k(n)} q_n(\sigma r) d\sigma. \end{aligned}$$

The hypothesis of the theorem implies that  $u_n(R, t)$  and  $(\partial_r u_n)(R, t)$  are zero for  $R \leq t \leq T$ . So repeating the standard argument for one dimensional hyperbolic inverse problems with reflection data, as in [17], or repeating just the sideways energy argument in the proof of Theorem 1.3 without the complication of the angular terms, one may show that  $q_n(r) = 0$  for  $R \leq r \leq (R + T)/2$ .

---

<sup>2</sup>Use the definition of  $q_n$  and  $u_n$ , observe that the  $\phi_n(\theta)$  are eigenvalues of the spherical Laplacian, and use the Divergence Theorem on  $S$  to transfer the Laplacian from the  $\phi_n$  to  $q$  or  $u$  - see Theorems 2 and 4 in [14].

## 4 Proof of Theorem 1.6

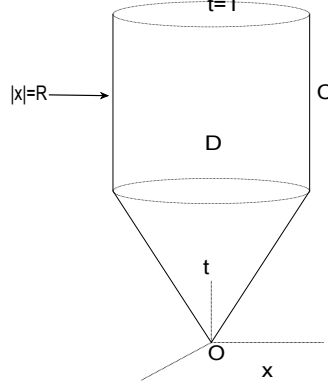


Figure 6: Transmission data problem

Let (see Figure 6)  $B$  denote the origin centered ball of radius  $R$  in  $\mathbb{R}^3$ ,  $D$  the region

$$D := \{(x, t) \in \mathbb{R}^3 \times \mathbb{R} : |x| \leq R, |x| \leq t \leq T\},$$

and as before  $C$  the cylinder

$$C := \{(x, t) : |x| = R, R \leq t \leq T\}.$$

Let  $u_i$ ,  $i = 1, 2$  be the solutions of (1.4), (1.5) when  $q = q_i$ ; define  $q = q_1 - q_2$  and  $u = u_1 - u_2$ . Then  $u$  satisfies

$$u_{tt} - \Delta u - q_1 u = q u_2, \quad (x, t) \in D \quad (4.1)$$

$$u(x, |x|) = \int_0^1 q(\sigma x) d\sigma. \quad (4.2)$$

Then, restricting attention to the cylindrical region  $B \times [R, T]$ , from [3] we have the following stability estimate for the time-like Cauchy problem (note  $T > 3R$ ): there is a constant  $C_1$  dependent only on  $M, R, T$  so that

$$\|u(\cdot, t)\|_{H^1(B)}^2 + \|u_t(\cdot, t)\|_{L^2(B)}^2 \leq C_1 \left( \|q u_2\|_{L^2(B \times [R, T])}^2 + \|u\|_{H^1(C)}^2 + \|u_r\|_{L^2(C)}^2 \right), \quad R \leq t \leq T. \quad (4.3)$$

Next, if we multiply (4.1) by  $u_t$  and use the techniques for standard energy estimates (backward in time) on the region  $|x| \leq t \leq R$ , we obtain

$$\int_B |\bar{u}(x)|^2 + |\nabla \bar{u}(x)|^2 dx \leq C_2 \left( \iint_{|x| \leq t \leq R} |q u_2|^2 dx dt + \|u(\cdot, R)\|_{H^1(B)}^2 + \|u_t(\cdot, R)\|_{L^2(B)}^2 \right) \quad (4.4)$$

where  $\bar{u}(x) = u(x, |x|)$  and  $C_2$  depends only on  $M, R$ . Hence, combining (4.3), (4.4) we obtain

$$\int_B |\bar{u}(x)|^2 + |\nabla \bar{u}(x)|^2 dx \leq C_3 \left( \|q u_2\|_{L^2(D)}^2 + \|u\|_{H^1(C)}^2 + \|u_r\|_{L^2(C)}^2 \right) \quad (4.5)$$



where  $C_3$  depends only on  $R, T, M$ . Now  $r\bar{u}(x) = \int_0^r q(s\theta) ds$ , hence  $q(x) = (r\bar{u})_r = \bar{u} + r\bar{u}_r$ . So

$$q^2 \leq 2(\bar{u}^2 + r^2\bar{u}_r^2) \leq 2\max(1, R^2)(\bar{u}^2 + \bar{u}_r^2) \leq 2\max(1, R^2)(\bar{u}^2 + |\nabla\bar{u}|^2),$$

and

$$\|q\|_{L^2(B)}^2 \leq C_4 \left( \|qu_2\|_{L^2(D)}^2 + \|u\|_{H^1(C)}^2 + \|u_r\|_{L^2(C)}^2 \right) \quad (4.6)$$

with  $C_4$  dependent only on  $R, T, M$ . Finally, using Theorem 1.1, we have

$$\|qu_2\|_{L^2(D)} \leq \|u_2\|_{L^\infty(D)} \|q\|_{L^2(D)} \leq \mathcal{N}(T, \|q_2\|_\infty) \|q\|_{L^2(D)}$$

where the  $\|q_2\|_\infty$  norm is over the region  $|x| \leq (R+T)/2$ . Since  $\mathcal{N}(T, \|q_2\|_\infty)$  goes to zero as  $\|q_2\|_\infty$  approaches 0, we can choose a  $\delta > 0$  so that

$$C_4\mathcal{N}(T, \|q_2\|_\infty) < \frac{1}{2}$$

if  $\|q_2\|_\infty \leq \delta$ ; note that this  $\delta$  will depend only on  $R, T, M$ . Using this in (4.6), we conclude that if  $\|q_2\|_\infty \leq \delta$  then

$$\|q\|_{L^2(B)}^2 \leq C_5 \left( \|u\|_{H^1(C)}^2 + \|u_r\|_{L^2(C)}^2 \right) \quad (4.7)$$

with  $C_5$  dependent only on  $R, T, M$ .

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